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*Comment*

# Comment on "Geodesics in Rotating Systems"

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Ashworth and Davies (1977) have recently discussed space-time geodesics and spatial geodesics on a rotating disk. By hypothesis, timelike geodesics in space-time represent possible paths of free particles, null geodesics in space-time represent possible paths of photons, and spatial geodesics are defined as paths along which the distance between any two fixed points is a minimum.

A special feature of their work is that the geodesic solutions are expressed in terms of two different coordinate systems $S_1(r_1, \theta_1, z_1, t_1)$ and $S_2(r_2, \theta_2, z_2, t_2)$, each of which is associated with the rotating disk. According to Ashworth and Davies, the two solutions $\theta_1 = \theta_1(r_1)$ and $\theta_2 = \theta_2(r_2)$, for each type of geodesic considered, are fully equivalent, the distinction between them being related to the different types of measurement which observers on the disk would be expected to make. However, it will be shown in this note that $\theta_1$, unlike $\theta_2$, does not have the status of a coordinate in the normal sense (one of four variables which serve to label events in space-time) and that the solutions quoted are therefore not equivalent. We argue nevertheless that $\theta_1$ does possess some significance, i.e., it is a plane polar coordinate in a particular type of *representation* (on a Euclidean plane) of geodesic paths on the disk.

Consider a disk which rotates with constant angular velocity $\omega$ with respect to an inertial frame, and assume space-time is flat. Using the notation of Ashworth and Davies (except that we suppress the $z$ coordinate throughout), let $(r, \theta, t)$ and $(r_2, \theta_2, t_2)$ be coordinates in the inertial and disk frames, respectively, related by the Galilean transformation

$$r = r_2, \qquad \theta = \theta_2 + \omega t_2, \qquad t = t_2 \tag{1}$$

the axis of rotation passing through $r = r_2 = 0$. Application of the usual relativistic postulates (Møller, 1952; Landau and Lifshitz, 1971) to the metric

$$\begin{aligned} ds^2 &= dr^2 + r^2\, d\theta^2 - c^2\, dt^2 \\ &= dr_2^{\,2} + r_2^{\,2}\, d\theta_2^{\,2} + 2\omega r_2^{\,2}\, d\theta_2\, dt_2 - (c^2 - \omega^2 r_2^{\,2})\, dt_2^{\,2} \end{aligned} \tag{2}$$

leads to the conclusion that the distance between the reference points $(r_2, \theta_2)$ and $(r_2 + dr_2, \theta_2 + d\theta_2)$ on the disk is

$$d\sigma_2 = \left( dr_2^{\,2} + \frac{r_2^{\,2}\, d\theta_2^{\,2}}{1 - \omega^2 r_2^{\,2}/c^2} \right)^{1/2} \tag{3}$$

In variational terms, space-time geodesics and spatial geodesics are described by $\delta \int_A^B ds = 0$ and $\delta \int_A^B d\sigma_2 = 0$, respectively.

Ashworth and Davies first find the explicit solution for a space-time geodesic in the $(r_2, \theta_2, t_2)$ coordinates, eliminating $t_2$ in the process so that the final result, $\theta_2 = \theta_2(r_2)$, is the locus of points traced out by the moving particle. The solution involves a parameter $u$, which is the (constant) velocity of the particle in the inertial frame. For a null geodesic, the appropriate result is obtained simply by setting $u = c$. They then proceed to show how the same result may be obtained by the method of "instantaneous Lorentz frames," essentially a special-relativistic method involving relationships between neighboring inertial frames whose origins are momentarily at rest with respect to the disk.

It is during the latter analysis that the arguments of Ashworth and Davies become questionable. They introduce "a system of cylindrical coordinates $S_1(r_1, \theta_1, z_1, t_1)$" but do not at first specify what these coordinates are, apart from indicating that they are somehow associated with the disk. Later it becomes clear that $r_1$ is identical to $r_2$ (and $t_1$ is presumably identical to $t_2$), but that $d\theta_1$ is related to $d\theta_2$ by

$$d\theta_1 = d\theta_2 (1 - \omega^2 r_2^{\,2}/c^2)^{-1/2} \tag{4}$$

It is now obvious, however, that the differential $d\theta_1$ is inexact, i.e., the transformation from $\theta_2$ to $\theta_1$ is nonholonomic. Hence $\theta_1$ has no general meaning which is independent of a particular path of integration. Expressed in another way, there is no 1:1 correspondence between reference points on the disk and coordinates $(r_1, \theta_1)$, whereas there is such a correspondence for the coordinates $(r_2, \theta_2)$. It cannot therefore be maintained that "the systems $S_1$ and $S_2$ are equally valid for describing events in a rotating system. . . ."

Despite the objection that $\theta_1$ has no path-independent meaning, Ashworth and Davies show in a diagram (Figure 2 of their paper) part of the path of a particle which apparently moves from $(r_1 + dr_1, \theta_1)$ to $(r_1, \theta_1 + d\theta_1)$. They calculate by the method of instantaneous Lorentz frames the angle $\phi$ between the direction of motion of the particle and the radial line from the origin to $(r_1 + dr_1, \theta_1)$, and then equate $\tan\phi$ to $r_1\, d\theta_1/dr_1$. On integration, the "solution" $\theta_1 = \theta_1(r_1)$ is obtained for the geodesic. Treating $r_1$ and $\theta_1$ as conventional polar coordinates on a Euclidean plane, this equation is found, in the case of a null geodesic, to represent a circular arc, a result which is taken as confirmation of the previous work of Ashworth and Jennison (1976).

We now indicate how in our view the defects in this analysis can be corrected, and we shed light on the significance of the variable $\theta_1$. Adopting Ashworth and Davies's expression for $\tan\phi$, we consider a particle which moves on the disk between the reference points $(r_2, \theta_2)$ and $(r_2 + dr_2, \theta_2 + d\theta_2)$. Noting from equation (3) that in doing so the particle moves a distance $dr_2$ in the radial direction and $r_2\, d\theta_2(1 - \omega^2 r_2{}^2/c^2)^{-1/2}$ in the circuital direction, we equate $\tan\phi$ to $r_2(1 - \omega^2 r_2{}^2/c^2)^{-1/2}\, d\theta_2/dr_2$. On integration we immediately recover the previously derived solution $\theta_2 = \theta_2(r_2)$ for the locus of points traced on the disk by the moving particle. (It may be worth noting however, that a much simpler method for finding this solution consists in writing down the corresponding solution in terms of the inertial frame coordinates $(r, \theta, t)$, transforming to the rotating frame coordinates $(r_2, \theta_2, t_2)$ by equation (1), then eliminating the time coordinate.)

On the other hand it is perfectly permissible to define $r_1 = r_2$ and $d\theta_1 = d\theta_2(1 - \omega^2 r_2{}^2/c^2)^{-1/2}$ so that $\tan\phi$ can be correctly equated to $r_1\, d\theta_1/dr_1$; integration then yields the result $\theta_1 = \theta_1(r_1)$ found by Ashworth and Davies. But the question now arises: what is the meaning of the variable $\theta_1$? As we have already noted, $\theta_1$ cannot serve to label reference points on the disk, since its value at any point on the disk would depend on the path of integration selected. However, we observe that $\tan\phi = r_1\, d\theta_1/dr_1$ correctly relates infinitesimal distances traveled in the radial and circuital directions to the instantaneous direction of motion for a trajectory on a Euclidean plane whose polar coordinates are $(r_1, \theta_1)$. Hence integration of that equation must yield a *mapping* of the disk trajectory into a Euclidean plane with polar coordinates $(r_1, \theta_1)$, a mapping which correctly displays the instantaneous direction of motion of the particle or photon as a function of distance from the origin, but which gives a misleading impression of other aspects of the disk trajectory. In contrast, a mapping $(r_2 \to r_2' = r_2, \theta_2 \to \theta_2' = \theta_2)$ of the disk coordinates into a Euclidean plane with polar coordinates $(r_2', \theta_2')$ gives a different representation of the trajectory, one which correctly portrays the relationship between $r_2$ and $\theta_2$ for the trajectory but which gives a misleading impression of the instantaneous direction of motion and of distance traveled. The distinction between the two mappings arises simply because the intrinsic geometry on the disk is non-Euclidean. These conclusions are in agreement with those reported elsewhere by McFarlane and McGill (1978) and obviously apply to spatial geodesics as well as to space-time geodesics. It should be noted, however, that the two types of mapping are not quite on an equal footing, since in the first case we can speak only of trajectories (on the disk) being mapped onto other trajectories (on a Euclidean plane), whereas in the second case the whole disk is mapped into a Euclidean plane. This means that greater caution must be exercised in interpreting diagrams of the first type: for example, the point of intersection of two trajectories on a diagram of the first type corresponds in general not to one reference point on the disk but to two such points.

Finally we note that Ashworth and Davies explain the distinction between the two coordinate systems $S_1$ and $S_2$ in terms of difference in experimental procedure. They associate the description of the motion in terms of $(r_2, \theta_2, t_2)$ with measurements made by an observer at the center of rotation, and associate the description in terms of $(r_1, \theta_1, t_1)$ with "the interpretation of events made by an observer who is in synchronous rotation with the system and who actually moves through it, making measurements as he or she goes." However, these associations seem to us unconvincing and, in so far as they introduce an unnecessary subjective element, undesirable. In particular, we cannot see any reason why measurements on free particles or photons in terms of $(r_2, \theta_2, t_2)$ need be confined to observations carried out by an observer at the origin, nor do we understand why it need be assumed that the only measurements which observers distributed over the disk are likely to make are measurements of the local direction of a spatial path. It is of course desirable that one should be able to explain how, at least in principle, paths of free particles and photons on the disk could be determined experimentally so that they could be compared with theoretical predictions, and it is fairly easy to think of a number of ways in which this could be done. However, we do not subscribe to the view that theories can be properly interpreted or understood *only* in terms of experimental measurements, especially when in practice one particular type of measurement appears to be given excessive preference over others.

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